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20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*) In this report we studied the spectral theory of the operator that is induced from problems in the n-dimensional Euclidean space (either in the exterior domain or in the whole space) for the hyperbolic linear partial-differential equations with the generalized Neumann boundary condition. The resulting theory provides a foundation for studying the wave operator and scattering operator involved in scattering theory and possibly also for studying the respective inverse problem.		

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CONTENTS

SCATTERING THEORY FOR THE ACOUSTIC WAVE EQUATION IN AN ARBITRARY EXTERIOR DOMAIN

INTRODUCTION

The following is a preliminary report on some recent theoretical investigations pertaining to scattering theory. Further study and the physical implications of these results will be discussed elsewhere. The scheme for direct scattering theory is outlined first, and an important topic is indicated for the inverse problem.

If $u(x)$ is the difference between the instantaneous pressure and the equilibrium pressure, $\rho(x)$ is the equilibrium density of the medium, $c(x)$ is the local speed of sound, Ω is an open connected subset of R^n with bounded complement, D_t denotes $\partial/\partial t$, and $A \equiv c^2(x)\rho(x)\nabla \cdot 1/\rho(x)\nabla$, then the acoustic wave equation is

$$D_t^2 u = - Au, \qquad t \in R,\ x \in \Omega, \tag{1}$$

with the initial conditions

$$u(0,x) = f(x) \text{ and } D_t u(0,x) = g(x), \qquad x \in \Omega \tag{2}$$

and the *generalized Neumann boundary condition*

$$\int c^{-2}(x)\rho^{-1}(x)\mathrm{A}u(x)v(x)\,dx = \int \rho^{-1}(x)\nabla u(x) \cdot \nabla v(x)\,dx,\ x \in \Omega \text{ and } v, \nabla v \in L^2(\Omega). \tag{3}$$

If the domain Ω considered here has a smooth boundary $\partial\Omega$, say C^2, then boundary condition (3) is equivalent to the *classical Neumann boundary condition*

$$\nu \cdot \nabla u(t,x) = 0, \qquad x \in \partial\Omega, \tag{4}$$

with ν denoting the outward unit normal of $\partial\Omega$ at x.

This acoustic wave propagation problem is considered here as a perturbed system in contrast to the following, named "the unperturbed system":

$$\begin{aligned} D_t^2 v(t,x) &= - A_0 v(t,x) \\ &= - \nabla^2 v(t,x), \qquad x \in \Omega, \end{aligned} \tag{5}$$

Note: Manuscript submitted June 14, 1976.

with the same initial and boundary conditions (2) and (3) of the perturbed problem.

We impose here the general assumptions that apply throughout this report:

Assumption 1. The exterior domain Ω has the *finite tiling property:* there exist an open set O in R^n, compact sets $K_1, \ldots, K_N$ in R^n, and nonzero vectors $x^{(1)}, \ldots, x^{(N)}$ such that

$$\partial\Omega \subset O, \tag{6}$$

$$O \cap \Omega \subset UK_j, \qquad 1 \leqslant j \leqslant N, \tag{7}$$

and

$$\{x = x_0 + tx^{(j)}: 0 < t < 1\} \subset \Omega, \qquad x_0 \in \Omega \cap K_j. \tag{8}$$

This property of an exterior domain is due to Wilcox [1]. Here we would not exclude the case $\Omega = R^n$.

Assumption 2. The density function $\rho(x)$ is $C^2(\Omega)$ and real valued, and for some constant $J > 1$

$$J \geqslant \rho(x) \geqslant J^{-1}, \qquad x \in \Omega. \tag{9}$$

Also,

$$\rho(x) \to 1 \text{ when } |x| \to \infty \tag{10}$$

and

$$D^\alpha \frac{1}{\rho(x)} \text{ behaves like } o(|x|^{-1}) \text{ when } |x| \to \infty \tag{11}$$

for all α, $1 \leqslant |\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant 2$.

Assumption 3. The local speed $c(x)$ is $C^1(\Omega)$ and real valued, and for some constant $K > 1$

$$K \geqslant c(x) \geqslant K^{-1}, \qquad x \in \Omega. \tag{12}$$

Also,

$$c(x) \to 1 \text{ when } |x| \to \infty \tag{13}$$

and

$$\nabla \, ln \; c(x) = o(|x|^{-1}) \text{ when } |x| \to \infty. \tag{14}$$

Assumption 4. The "Stummel condition" is satisfied by $q(x) \equiv c^{-2}(x)\rho^{-1}(x)$; that is, for some $a > 0$,

$$sup \int |q(y)|^2 |x-y|^{-n+4-a} \, dy < +\infty, \text{ if } n \geqslant 4, \tag{15}$$

where the integration variable y runs in the disk $\{|x-y| < 1\} \cap \Omega$ and the supremum is taken on $x \in \Omega$.

The differential operator A_0 defined by (5) on Ω, with assumption 1, subject to the generalized Neumann boundary condition, is a self-adjoint, nonnegative operator in the Hilbert space $L_2(\Omega)$. Its spectrum is the closed interval $[0, \infty)$ and is absolutely (spectral) continuous and without eigenvalues. These interesting results are proved by Wilcox [1]. With these four general assumptions we will show that the operator A defined by (1) subject to the generalized Neumann boundary condition is also a self-adjoint, nonnegative operator in the same Hilbert space $L_2(\Omega)$ and that its spectrum contains the interval $(0, \infty)$, is contained in $[0, \infty)$, and is absolutely continuous. The only uncertainty occurs when the origin is to be an eigenvalue. These results are discussed in the next section.

The scattering operator S is unitary if the Møller wave operators $W_\pm$ are orthogonal,

$$W_\pm^* W_\pm = I, \tag{16}$$

and are complete,

$$W_\pm W_\pm^* = I - E(0+), \tag{17}$$

where $E(\lambda)$ is the resolution of the identity for A. Therefore, in the third section, we will discuss the existence of the Møller wave operators $W_\pm$ and properties (16) and (17).

The inverse problem of the scattering theory is to construct $c(x)$ and $\rho(x)$ from the "scattering amplitude." The study of the one-dimensional plasma inverse problem by Szu, Yang, Ahn, and Carroll [2] and the articles by Faddeev [3] and Newton [4] on the three-dimensional inverse scattering problem for the Schrödinger equation all center on discussion of the integral equation

$$u(x,k;w) = e^{ix \cdot k} + \int G(x-y,k;w) A' u(y,k;w) \, \mathrm{dy} \tag{18}$$

for all y such that $y \cdot w > x \cdot w$ for each unit vector w in R, where $A' = A - A_0$ and where $G(x,k;w)$ is a certain class of Green functions of the reduced equation from the unperturbed system:

$$(A_0 + k^2) G(x,k) = \delta(x). \tag{19}$$

This particular problem is connected with the direct problem and is discussed in the fourth section.

Other references relating to the results can be obtained through those cited in this report.

SOLUTIONS OF THE ACOUSTIC WAVE EQUATION WITH THE NEUMANN BOUNDARY CONDITION AND ITS SPECTRAL THEORY

A solution of the mixed initial-boundary-value problem of Eqs. (1) through (3) will be constructed by using a famous spectral theorem. Then the absolute continuity of the spectrum of A will be is studied. The resulting theory provides a preparation for constructing the wave operator and scattering operator.

We recall here the initial-boundary-value problem:

$$D_t^2 u = - \mathrm{Au},\ t > 0,$$

$$\equiv c^2(x)\rho(x)\nabla \cdot 1/\rho(x)\nabla u, \qquad x \in \Omega, \tag{20}$$

$$u(0, x) = f(x) \text{ and } D_t u(0, x) = g(x), \qquad x \in \Omega; \tag{21}$$

$$\int c^{-2}(x)\rho^{-1}(x)\mathrm{Au}(x)v(x)\ \mathrm{dx} = \int \rho^{-1}(x)\nabla u(x)\cdot\nabla v(x)\ \mathrm{dx},\ x \in \Omega \text{ and } v,\nabla v \in L^2(\Omega). \tag{22}$$

The formulation of the problem will based on the following function spaces:

$$u \in L_2(\Omega) \Leftrightarrow \int |u(x)|^2\ \mathrm{dx} < \infty, \qquad x \in \Omega, \text{ in the Lebesque measure,} \tag{23}$$

$$u \in H_m(\Omega) \Leftrightarrow D^\alpha u \in L_2(\Omega), \qquad |\alpha| \leqslant m, \tag{24}$$

$$u \in L_2(\Omega;c^2\rho) \Leftrightarrow (c^2\rho)^{-1/2}u \in L_2(\Omega), \tag{25}$$

$$u \in H_m(\Omega;c^2\rho) \Leftrightarrow (c^2\rho)^{-1/2}D^\alpha u \in L_2(\Omega), \qquad |\alpha| \leqslant m, \tag{26}$$

and

$$u \in X_i \Leftrightarrow \rho^{1/2}\nabla^j u \in L^2(\Omega). \tag{27}$$

These spaces are Hilbert spaces with respect to the following inner products respectively:

$$(u,v) = \int_\Omega u(x)v(x)\ \mathrm{dx}, \qquad x \in \Omega, \tag{28}$$

$$(u,v)_m = \sum (D^\alpha u, D^\alpha v), \qquad |\alpha| \leqslant m, \tag{29}$$

$$\langle u,v\rangle = \int c^{-2}(x)\rho^{-1}(x)\ u(x)\ v(x)\ \mathrm{dx}, \qquad x \in \Omega, \tag{30}$$

$$\langle u,v\rangle_m = \sum \langle D^\alpha u, D^\alpha v\rangle,\ |\alpha| \leqslant m, \tag{31}$$

$$[u,v]_i = \sum \int \frac{1}{\rho(x)}\nabla^j u(x)\nabla^j \mathrm{v}(\mathrm{x})\ \mathrm{dx}, \qquad x \in \Omega \text{ and } 1 \leqslant j \leqslant i \leqslant 2. \tag{32}$$

It is clear from the characteristics (9) and (12) of $\rho(x)$ and $c(x)$ that $L_2(\Omega) = H_0(\Omega)$ (which is equivalent to $L_2(\Omega;\, c^2\rho) = H_0(\Omega;\, c^2\rho)$, in the view of their norms).

Suppose that the boundary $\partial\Omega$ of Ω is sufficiently smooth, say C^2. Then Green's theorem implies that

$$\begin{aligned}\langle \mathrm{Au}, v\rangle &= -\int\left[\nabla \cdot \frac{1}{\rho(x)}\nabla u(x)\right] v(x)\ \mathrm{dx}\\ &= \int \frac{1}{\rho(x)}\nabla u(x) \cdot \nabla \mathrm{v}(x)\ \mathrm{dx}\\ &= \int\left[\nu \cdot \frac{1}{\rho(x)}\nabla \rho u(x)\right] v(x)\ dS(x), \qquad x \in \partial\Omega, \end{aligned} \tag{33}$$

where v is the unit outward normal of $\partial\Omega$ at x. This means that

$$-\langle \mathrm{Au}, v\rangle + [u,v]_1 = \int \frac{1}{\rho(x)}[\nu \cdot \nabla u(x)] \cdot v(x)\, dS(x), \qquad x \in \partial\Omega. \tag{34}$$

Therefore, for $u \in X_2$, u satisfies the classical Neumann boundary condition $\nu \cdot \nabla u(t,x) = 0$ if and only if ν satisfies the relation (22), which in the new notation is

$$\langle \mathrm{Au}, v\rangle = [u,v]_1, \qquad v \in X_1 \cap L_2(\Omega; c^2\rho). \tag{35}$$

Definition 1. A function $u \in X_2$ is said to satisfy the *generalized Neumann condition* if and only if (35) holds.

This definition does not require the assumption of the smoothness of the boundary $\partial\Omega$ of Ω. However it defines the classical Neumann condition if $\partial\Omega$ is smooth.

Furthermore a definition is introduced for a closed subspace in X_2:

$$u \in \mathcal{H} \Leftrightarrow u \in X_2 \text{ and satisfies (35).} \tag{36}$$

The construction of a solution of the initial-boundary-value problem is based on the linear operator A in H_2 given by (20) with domain

$$D(A) = \mathcal{H}. \tag{37}$$

Theorem 1. A *is a self-adjoint operator on the Hilbert space* $L_2(\Omega; c^2\rho)$. *Moreover* $A \geqslant 0$.

The verification of the assertions is based on the following result:

Lemma 1. Let $\mathcal{H}$ *be a Hilbert space and let* $L\colon \mathcal{H} \to \mathcal{H}$ *be a linear operator densely defined in* $\mathcal{H}$. *Assume that* $L \subset L^*$, *the adjoint of* L, $L \geqslant 0$, *and that the range* $R(I + L)$ *of* $I + L$ *is* $\mathcal{H}$. *Then* L *is self-adjoint.*

Proof of Lemma 1. $L \geqslant 0$ indicates that the deficiency indices of the symmetric operator L and of L^*, an extension of L, are equal [5,p.268]. Condition $R(I{+}L) = \mathcal{H}$ implies the deficiency index is zero and hence L is self-adjoint.

Proof of Theorem 1. Verify the conditions of lemma 1 for $\mathcal{H} = L_2(\Omega;c^2\rho)$ and $L = A$. The space $C_0(\Omega)$ of infinitely-many-times continuously differentiable functions with compact support in Ω is a subset of $D(A) = \mathcal{H}$; hence $D(A)$ is dense in $L_2(\Omega;c^2\rho)$. Let u and v be any two elements of $D(A)$. Then (35) and (32) yield

$$\langle Au, v\rangle = [u, v]_1 = \int \frac{1}{\rho(x)} \nabla u(x) \cdot \nabla v(x)\, dx = \langle u, Av\rangle. \tag{38}$$

Consequently $D(A) \subset D(A^*)$ and $A^*u = Au$ for all $u \in D(A)$; that is, $A \subset A^*$.

The assertion $A \geqslant 0$, also one of the conditions to be checked, follows from the result yielded by (35) and (32) that for all $u \in D(A)$

$$\langle Au, u\rangle = [u, u]_1 = \int |\rho^{-1/2}(x)\nabla u(x)|^2\, dx, \quad x \in \Omega. \tag{39}$$

The only condition left to be verified, $R(I+A) = L_2(\Omega;c^2\rho)$, means that for each f in $L_2(\Omega;c^2\rho)$ there exists an element u of $D(A)$ such that

$$\langle u, v\rangle + \langle Au, v\rangle = \langle f, v\rangle \text{ for all } v \text{ in } H_1(\Omega;c^2\rho). \tag{40}$$

This, together with (35), is equivalent to

$$\langle u, v\rangle + [u, v]_1 = \langle f, v\rangle \text{ for all } v \text{ in } H_1(\Omega;c^2\rho). \tag{41}$$

Sufficiently, if the equivalent inner product for $H_1(\Omega;c^2\rho)$ that we use is

$$\{u, v\} = \langle u, v\rangle + [u, v]_1, \qquad u, v \in H_1(\Omega;c^2\rho), \tag{42}$$

we need to verify the existence of u such that

$$\{u, v\} = \langle f, v\rangle, \qquad v \in H_1(\Omega;c^2\rho). \tag{43}$$

However,

$$|\langle f, v\rangle| \leqslant \langle f, f\rangle^{1/2}\langle v, v\rangle^{1/2} \leqslant \text{const } \{f, f\}^{1/2}\{v, v\}^{1/2}.$$

The Riesz representation theorem in the Hilbert space $(H_1(\Omega;c^2\rho), \{. , .\})$ yields the existence of an element u in $H_1(\Omega;c^2\rho)$ satisfying (43) and then (41). On the other hand (41) implies (40) for all v in $c_0^\infty(\Omega)$. Thus $Au = f - u$ with the member on the right side in $L_2(\Omega,c^2\rho)$; hence Au is also in $L_2(\Omega,c^2\rho)$. Moreover, because the validity of (40) itself, is implied when, $c_0^\infty(\Omega)$ is dense in $H_1(\Omega,c^2\rho)$, the combination of (40) and (41) ensures that u satisfies the generalized Neumann condition (35). Thus $R(I+A) = \mathcal{H}$ is verified, and the proof of theorem 1 is complete.

Therefore the Kato [5,p.331] second representation theorem ensures the following corollary.

Corollary 1. A has a nonnegative square root $A^{1/2}$ whose domain $D(A^{1/2}) = H_1(\Omega;c\rho^{1/2})$ has the inner product

$$\{u, v\} = \sum \int c^{-1}(x)\rho^{-1/2}(x)D^{\alpha}u(x)D^{\alpha}v(x)\, dx, \qquad x \in \Omega, |\alpha| \leqslant 1. \tag{44}$$

Furthermore $A^{1/2}$ satisfies the relation

$$\{A^{1/2}u, A^{1/2}u\} = \sum (\rho^{-1/2}(x)D_j u, \rho^{-1/2}D_j u), \qquad 1 \leqslant j \leqslant n. \tag{45}$$

From the results in theorem 1 and corollary 1, the argument of Wilcox [1], with slight adjustment, gives the following theorem.

Theorem 2. For each f in D(A) and g in $D(A^{1/2})$ there exists a uniquely defined strict solution u with finite energy of the initial-boundary-value problem (20), (21), *and* (35) *with $t \in R$ such that*

$$u \in C^2[R, L_2(\Omega,c^2\rho)] \cap C^1[R, H_1(\Omega,c\rho^{1/2})] \cap C(R, \mathcal{H}),$$

and u has the energy integral in the two equivalent forms

$$\begin{aligned} E(u, \Omega, t) &= \{D_t u(t), D_t u(t)\} + \sum \{D_j n(t), D_j u(t)\}, \qquad j = 1, \ldots, n, \\ &= \{D_t u(t), D_t u(t)\} + \{A^{1/2}u(t), A^{1/2}u(t)\}. \end{aligned} \tag{46}$$

and has the constancy of energy

$$E(u, \Omega, t) = \sum \{D_j f, D_j f\} + \{g, g\}. \tag{47}$$

From the spectral theorem for A and the associated operator calculus, we have the following theorem.

Theorem 3. For real-valued functions f in $L_2(\Omega)$ and g in $D(A^{-1/2})$, define

$$h = f + iA^{-1/2}g \in L_2(\Omega). \tag{48}$$

Then the solution in $L_2(\Omega)$ defined by

$$u(t) = (\cos tA^{1/2})f + (A^{-1/2} \sin tA^{1/2})g \tag{49}$$

with bounded coefficient operators satisfies

$$U(t, x) = Re\, v(t, x), \tag{50}$$

where $v(t, x)$ is the complex-valued solution in $L_2(\Omega)$ defined by

$$v(t, 0) = e^{-itA^{-1/2}}h. \tag{51}$$

Because the operator A is nonnegative, its spectrum $\sigma(A)$ is contained in the interval $[0,\infty)$. The nonexistence of the positive eigenvalue is ensured by the following theorem.

Theorem 4 (Mochizuki [6]. *Assume assumptions 2 through 4. Then, $u = 0$ in Ω is the only $L_2(\Omega)$ solution of the equation*

$$\nabla \cdot \frac{1}{\rho(x)} \nabla u + \lambda c^{-2}(x)\rho^{-1}(x)u = 0, \qquad x \in \Omega. \tag{52}$$

Proof. It suffices to check the conditions imposed by Mochizuki. His first three conditions are presented by assumptions 2 through 4 by setting $a_{jk}(x) = \sqrt{jk}\,\rho^{-1}(x)$, $b_j = 0$, and $q(x) = \lambda c^{-2}(x)\rho^{-1}(x)$. Particularly, if $n \leqslant 3$, the "Stummel condition"

$$\sup\int |q(y)|^2\, dy < \infty, \qquad (y \in \Omega\colon |x-y| < 1, x \in \Omega) \tag{53}$$

is a consequence of the boundedness and the smoothness of $c(x)$ and $\rho(x)$ on Ω. His fourth condition is given by his remark 1.2 and by (9), (11), and (14). As for the unique continuation property in his last condition, the smoothness assumption on $c(x)$ and $\rho(x)$ implies the Hölder condition [7, p. 23], which gives the property (according to his remark 1.1, or also according to Ref. 8 or Ref. 9).

Subject to some adjustment, the employment of the standard principle of limiting absorption yields the absolute continuity of the continuous spectrum of A. The detailed proof will appear in a follow-up article. The precise statement is the following.

Theorem 5. *The resolution $E(s)$ of the identity for the operator A is absolutely continuous on any closed interval in* $(0, \infty)$.

Remark 1. Whether $\lambda = 0$ is an eigenvalue or belong to the continuous spectrum is not yet clear.

DISCUSSION OF THE MØLLER WAVE OPERATORS

As indicated in the Introduction, all properties of A studied in the last section hold for A_0. Moreover the spectrum of A_0 is $[0,\infty)$ and is absolutely continuous. These are results of Wilcox [1]. These results and those of the last section ensure the existence of the unitary groups $e^{-iL_0 t}$ and e^{-iLt}, $-\infty < t < \infty$, associated with the self-adjoint operators $L_0 = A_0^{1/2}$ and $L = A^{1/2}$.

The strong limits

$$W_\pm = S-\lim e^{iLt}e^{-iL_0 t}, \qquad t \to \pm\infty \tag{54}$$

are called the Møller wave operators. This yields

$$\lim \|e^{-iLt}h - e^{-iL_0 t}h_\pm\| = 0, \qquad t \to \pm\infty, \tag{55}$$

and

$$h = W_{\pm} h_{\pm}. \tag{56}$$

The following map is the scattering operator:

$$S: f_- \to f_+ = Sf_-. \tag{57}$$

It is required to be unitary on $L_2(\Omega)$, which is an easy consequence of the orthogonality (16) and the completeness (17) of the Møller wave operators.

The existence of the strong limit (54) is proved by employing the vanishing, when $t \to \pm\infty$, of the local energy of the solution to the unperturbed system and employing the decaying in rate $-1 - \delta$ of $|t|$ for the first second-order derivative of the solution to the unperturbed system.

The coincidence of the Møller wave operator $W_{\pm}$ with the stationary wave operators $U_{\pm}$ and the properties of the orthogonality and the completeness for $U_{\pm}$ guarantee the corresponding properties for $W_{\pm}$. The proof of the coincidence and of these two properties for $U_{\pm}$ is based on the expansion principle of the generalized eigenfunctions of A, which has been employed by a dozen different authors, including Ikebe [10], Mochizuki [11,12], and Wilcox [1]. The details will be given in the follow-up article. A portion of the article involves the principle of limiting absorption, which is studied in the next section.

PRINCIPLE OF LIMITING ABSORPTION AND THE INVERSE PROBLEM

There is no doubt about the role played by the principle of limiting absorption in the eigenfunction expansion, which is a keystone for scattering theory. However, in the three-dimensional inverse scattering theory for the Schrödinger equation, both Faddeev [3] and Newton [4] studied the "Volterra" integral equation, where the principle of limiting absorption again played an essential role. We will apply the principle of limiting absorption to the acoustic wave equation. This should bring some light to the study of the inverse problem of the scattering theory for the acoustic wave in three dimensions, which will extend Ref. 2.

In the investigation of recovering the perturbed operator $A' = A - A_0$, the reduced wave equation of acoustics without an obstacle,

$$Au(x,k) + k^2 u(x,k) = 0, \qquad x, k \in R^3, \tag{58}$$

is expected to have a set of solutions $u(x, k; w)$ such that the function $u(x, k; w)e^{-ik\cdot x}$ has an analytic continuation into an upper halfplane in the variable $s = k^{\perp} \cdot \gamma$ at a fixed x and $k^{\perp} = k - (k\cdot\gamma)\gamma$; and $u(x, k; w)e^{-ik\cdot x} - 1$ decreases with large $|s|$. From an alternative form of (58), namely,

$$A_0 u + k^2 u = A_1 u$$

$$\equiv k^2[1 - c^2(x)\rho^2(x)]u - \frac{\nabla\rho(x)}{\rho(x)} \cdot \nabla u, \qquad k, x \in R^3, \tag{59}$$

we should study the integral equation

$$u(x,k;w) = e^{ik\cdot x} + \int G(x - y, k; w)A_1 u(y, k; w)\, dy \tag{60}$$

with x, k in R^3 and with a unit vector w, where the Green function $G(x, k; w)$ is a limiting value of the following function with $Im\ s = 0$:

$$G(x, \mu; s) = (2\pi)^{-3}\int e^{i(m+sw)\cdot x}[(m + sw)^2 - \mu^2 - s^2]^{-1}\, dm, \tag{61}$$

with $m \in R^3$, $w = q/|q|$, $s = k\cdot w + i|q|$, and $\mu^2 = k - (k\cdot w)^2$. As indicated by Faddeev, the irregularities in the expression under the integral sign are proven by the equations

$$m^2 = \mu^2 \text{ and } m\cdot w = 0 \tag{62}$$

and do not depend on s with $Im\ s \neq 0$. This implies, through a singular Fourier integral, that the function defined by (61) is an analytical function of s in the upper half plane and that

$$|G(x, \mu; s) \exp\,[Im\ s(x\cdot w)]| \leqslant \text{const } t(1 + |x|)^{-1}. \tag{63}$$

The existence of an analytical continuation of the Green function $G(x, k; w)$ in the upper halfplane in the variable $s = k\cdot w$, which satisfies (63), is just the characteristic property that defines this function uniquely.

On the other hand, A_1 in (59) is still a differential operator. We remark that

$$G(x - y, k; w)\nabla u(y, k; w)\cdot \frac{\nabla\rho(y)}{\rho(y)}$$

$$= G(x - y; k; w)\left\{\nabla u(t, k; w)\frac{\nabla\rho(y)}{\rho(x)} - u(y, k; w)\left[\frac{\nabla^2\rho(x)}{\rho(x)} - \frac{|\nabla\rho(x)|^2}{\rho(x)}\right]\right\}$$

$$= \nabla_y\cdot\left[G(x - y, k; w)u(y, k; w)\frac{\nabla\rho(y)}{\rho(y)}\right] + \left\{\nabla_{x-y}G(x - y, k; w)\cdot\frac{\nabla\rho(y)}{\rho(y)}\right.$$

$$\left. - \left[\frac{\nabla^2\rho(x)}{\rho(x)} - \frac{|\nabla\rho(x)|^2}{\rho(x)}\right]G(x - y; k, w)\right\}u(y, k; w). \tag{64}$$

Formally this relation and the divergence theorem of Gauss give another representation of (60):

$$u(x, k; w) = e^{ik\cdot x} + \int G(x - y, k; w)A_2 u(y, k; w)\, dy$$

$$+ \int \nabla_{x-y}G(xpy, k; w)\cdot \frac{\nabla\rho(y)}{\rho(y)}\, u(y, k; w)\, dy, \tag{65}$$

where the multiplication operator A_2 is defined by

$$A_2 u = d^2[1 - c^2(y)\rho^2(y)] + \left[\frac{\nabla^2\rho(y)}{\rho(y)} - \frac{|\nabla\rho(y)|^2}{\rho(y)}\right]u(y, k; w). \tag{66}$$

Here the following condition is assumed:

$$\lim \int G(w - y, k; w)u(y, k; w)\tilde{y} \cdot \frac{\nabla\rho(y)}{\rho(y)}\, dS(y) = 0, \ |y| = r \to \infty, \tag{67}$$

with the surface integral over the sphere with radius r. However some $u(y, k; w)$ is required to be bounded, and since $G(w - y, k; w)$ has the asymptotic estimate (63), condition (11) in assumption 2 ensures (67). The representation (65) is not justified yet, because the asymptotic estimate for $\nabla_y G(y, k; w)$ at large $|y|$ is not clear.

This brings in the issue concerning the asymptotic estimate of both $G(y, k; w)$ and $\nabla_y G(y, k; w)$ given by (61) with the principle of limiting absorption employed.

In the rest of this section, we will follow the idea used in Ref. 13. First we recall a well-known formula [11, 13]:

$$\int_{-\infty}^{\infty} e^{irp}[r - (a + ib)]^{-1}\, dr = i\pi(\text{sign } b)H(p \text{ sign } b)e^{i(a+ib)p}. \tag{68}$$

where $H(t)$ is the Heaviside function. From (61) we have for a multiple index α

$$D_x^\alpha G(x, \mu; s) = (2\pi)^{-3}e^{isw\cdot x}\int_{-\infty}^{\infty} r^2(J_+ + J_-)\, dr, \tag{69}$$

and

$$\begin{aligned} J_+ + J_- = \int_{S^2} & [i(\tilde{m}r + sw)]^\alpha[\mu^2 + (s\tilde{m}\cdot w)^2]^{-1/2}e^{ir\tilde{m}\cdot x}\Big(\Big\{r + (s\tilde{m}\cdot w) \\ & +[\mu^2 + (s\tilde{m}\cdot w)^2]^{1/2}\Big\}^{-1} + \Big\{r + (s\tilde{m}\cdot w) - [\mu^2 \\ & +(s\tilde{m}\cdot w)^2]^{1/2}\Big\}^{-1}\Big)\, dS(\tilde{m}). \end{aligned} \tag{70}$$

Without loss of generality we can assume $\tilde{m}\cdot x = x_1\tilde{m}_1$. By the Morse transformation [14] we choose a local coordinate (η_1, η_2) at $(\pm 1,0,0)$ such that $\tilde{m}_1 = \pm(1 - \eta_1^2 - \eta_2^2)$. Then the estimate for large $|x|$ is

$$J_+ + J_- = \text{const}\,\frac{2\pi}{r|x|}\,[\mu^2 + (s\tilde{x}\cdot w)^2]^{-1/2}[e^{ir|x|}[i(\tilde{x}r + sw)]^\alpha(\{r + (s\tilde{x}\cdot w)$$

$$+ [\mu^2 + (s\tilde{x}\cdot w)^2]^{1/2}\}^{-1} + \{r + (s\tilde{x}\cdot w) - [\mu^2 + (s\tilde{x}\cdot w)^2]^{1/2})^{-1}\}$$

$$+ e^{-ir|x|}[i(-\tilde{x}r + sw)]^\alpha(\{r - (s\tilde{x}\cdot w) + [\mu^2 + (s\tilde{x}\cdot w)^2]^{1/2}\}^{-1}$$

$$+ \{r - (s\tilde{x}\cdot w) - [\mu^2 + (s\tilde{x}\cdot w)^2]^{1/2}\}^{-1})] + q(rx), \tag{71}$$

where $q(x)$ is estimated for large $|x|$ as

$$|q(x)| + |\nabla q(x)| \leqslant \text{const}\ |x|^{-2}. \tag{72}$$

Because the denominators in (70) would not vanish when $\tilde{m}\cdot w \neq 0$ (by (62)), there is no singularity in (71) when $\tilde{x}\cdot w \neq 0$. After substitution of (71) and (72) into (69) and then integration by parts, we have the estimation

$$D^\alpha G(x, \mu; s) \leqslant \text{const}\ |x|^{-2}, \qquad \text{if } \tilde{x}\cdot w \neq 0. \tag{73}$$

Assume now that $\tilde{x}\cdot w = 0$. Then (71) becomes

$$J_+ + J_- = \text{const}\ 2\pi(r|x|\mu)^{-1}\{e^{ir|x|}(i\tilde{x}rsw)^\alpha + e^{-ir|x|}[i(sw - \tilde{x}r)]^\alpha\}[(r + m)^{-1}$$

$$+ (r - m)^{-1}] + q(rx), \qquad (\text{if } \tilde{x}\cdot w = 0). \tag{74}$$

Let $\phi(r)$ be a C^∞ function with compact support such that $\phi(r)$ has values between 0 and 1 and is equal to 1 at $r = \pm\,\mu$. Therefore (69) and (74) imply that

$$D_x^\alpha G(x, \mu; s) = (2\pi)^{-3}\int_{-\infty}^{\infty} r^2(J_+ + J_-)[1 - \phi(r)]\ dr\ (\equiv I_1)$$

$$+ (2\pi)^{-2}(|x|\mu)^{-1}\ \text{const}\int_{-\infty}^{\infty}(e^{ir|x|}\phi_+(r; \tilde{x}, s, w, \alpha)$$

$$+ e^{-ir|x|}\phi_-(r; \tilde{x}, s, w, \alpha)([r + \mu]^{-1} + [r - \mu]^{-1})\ dr(\equiv I_2)$$

$$+ O(|x|^{-2}), \qquad \text{if } \tilde{x}\cdot w = 0. \tag{75}$$

where $\phi_\pm = r\phi(r)[i(sw \pm \tilde{x}r)]^\alpha$. There is no singularity in the integral for I_1. Integration by parts yields $I_1 = O(|x|^{-2})$. Employment of (68) for I_2 gives

$$I_2 = (2\pi|x|\mu)^{-1}\ \text{const}\ [\psi_+(|x|; \tilde{x}, s, w, \alpha)$$

$$+ \psi_-(-|x|; \tilde{x}, s, w, \alpha)] * [\sin(\mu|x|)H(|x|)], \tag{76}$$

where $f * g$ means the convolution of f and g and where the function $\psi_\pm(|x|; \tilde{x}, s, w, \alpha)$ is the Fourier transform of $\phi_\pm(r; \tilde{x}, s, w, \alpha)$ in r. Since $\phi_\pm$ is compactly supported in r, $\psi_\pm$ is a smooth rapidly decaying function in $|x|$. Therefore

$$D_x^\alpha G(x, \mu; s) = O[(|x|\mu)^{-1}], \qquad \text{for large } |x| \text{ with } \tilde{x} \cdot w = 0), \tag{77}$$

for any order α of derivatives.

Then for either $\tilde{x} \cdot w = 0$ or $\tilde{x} \cdot w \neq 0$, we have

$$|D_x^\alpha G(x, \mu; s) \exp \{Im\ s(x \cdot w)\}| \leqslant \text{const } (1 + |\mu x|)^{-1}, \qquad 0 \leqslant |\alpha| < \infty, \tag{78}$$

which is (63) when $\alpha = 0$.

This estimate ensures the two integrals on the right side of (65) are well-defined under the general assumptions for $c(x)$ and $\rho(x)$. Equation (65) will be further studied in the follow-up article.

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